Interactions and Focusing of Nonlinear Water Waves

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The coupled cubic nonlinear Schrödinger (CNLS) equations are used to study modulational instabilities of a pair of nonlinearly interacting two-dimensional waves in deep water. It has been shown that the full dynamics of these interacting waves gives rise to localized large-amplitude wavepackets (wave focusing). In this short letter we attempt to verify this result numerically using a Fourier spectral method for the CNLS equations.

I. INTRODUCTION

Extremely large size waves (commonly known as freak, rogue or giant waves) are very common in the open sea or ocean and they pose major hazard to mariners. As early as 1976, Peregrine [10] suggested that in the region of oceans where there is a strong current present, freak waves can form when action is conncentrated by reflection into a caustic region. A variable current acts analogously to filamentation instability in laser-plasma interactions [7, 8]. Freak waves are very steep and is a nonlinear phenomena, hence they cannot be represented and described by a linear water wave theory. Zakharov [18] in 2009 has noted that in the last stage of their evolution, their steepness becomes 'infinite', thereby forming a 'wall of water'. However, before such an instant in time, the steepness is higher than one for the limiting Stokes wave and before breaking the wave crest reaches three to four (sometimes even more) times higher than the crests of neighboring waves. The freak wave is preceded by a deep trough appearing as a 'hole in the sea'. On the other hand, a characteristic life time of a freak wave is short, typically ten of wave periods or so. For example, if the wave period is fifteen seconds, then their life time is just few minutes. Freak wave appears almost instantly from a relatively calm sea. It is, therefore, easy to appreciate that such peculiar features of freak waves cannot be explained by means of a linear theory. Even the focusing of ocean waves is a preconditions for formation of such waves.

It is now quite common to associate appearance of freak waves with the modulation instability of Stokes waves. This instability (known as the Benjamin–Feir instability) was first discovered by Lighthill [6] and the detail of theory was developed independently by Benjamin and Feir [1] and by Zakharov [14]. Zakharov showed slowly modulated weakly nonlinear Stokes wave can be described by nonlinear Shrödinger equation (NLSE) and that this equation is integrable [15] and is just the first term in the hierarchy of envelope equations describing packets of surface gravity waves. The second term in this hierarchy was calculated by Dysthe [3].

Since the pioneer work of Smith [12], many researchers attempted (both theoretically or numerically) to explain the freak wave formation by NLSE. Among diverse results obtained by them there is one important common observation which has been made by all, and that is, nonlinear development of modulational instability leads to concentration of wave energy in a small spatial region. This marks the possibility for formation of freak wave. Modulation instability leads to decomposition of initially homogeneous Stokes wave into a system of envelope solitons, or more strictly quasi-solitons [16, 17]. This state can be called 'solitonic turbulence', or 'quasisolitonic turbulence'.

In this letter, we consider the problem of a single solition in a homogeneous media, being subjected to modulational instability which eventually leads to formation of a system of soliton. We will show that the supercritical instability leads to maximum formation of soliton, concentrated in a small region. In going through subcritical instability the solitons coagulate to early stages of supercritical instability. Moreover, we investigate the full dynamics of nonlinearly interacting deep water waves subjected to modulational/filamentation instabilities, and we find that random perturbations can grow to form inherently nonlinear water wave structures, the so called freak waves, through the nonlinear interaction between two coupled water waves. The latter should be of interest for explaining recent

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observations in water wave dynamics.

II. NUMERICAL APPROACH

In a pioneering work, a theory for the modulational instability of a pair of two-dimensional nonlinearly coupled water waves in deep water, as well as the formation and dynamics of localized freak wave packets was presented [9, 11]. The two wave packets were investigated in the context of nonlinear optics by [19, 20], in Bise-Einsten condensates by [21, 22], in transmission by [23], and in plasmas by many other authors [24–28].

Following [11], we consider the two-dimensional CNLS equations in the following form

$$i\left(\frac{\partial A}{\partial t}+C_x\frac{\partial A}{\partial x}+C_y\frac{\partial A}{\partial y}\right)+\alpha\frac{\partial^2 A}{\partial x^2}+\beta\frac{\partial^2 A}{\partial y^2}+\gamma\frac{\partial^2 A}{\partial x\partial y}-\xi\,|A^2|A-2\zeta\,|B|^2A=0\,, \tag{1}$$

and

$$i\left(\frac{\partial B}{\partial t}+C_x\frac{\partial B}{\partial x}-C_y\frac{\partial B}{\partial y}\right)+\alpha\frac{\partial^2 B}{\partial x^2}+\beta\frac{\partial^2 B}{\partial y^2}-\gamma\frac{\partial^2 B}{\partial x\partial y}-\xi\,|B^2|B-2\zeta\,|A|^2B=0, \tag{2}$$

where A and B are the amplitudes of the slowly varying wave envelopes. The x and y components of the group velocity are given respectively by

$$C_x=\omega k/2\kappa^2 \quad \text{and} \quad C_y=\omega\ell/2\kappa^2$$

and the group velocity dispersion coefficients are

$$\alpha=\omega(2\ell^2-k^2)/8\kappa^4, \quad \beta=\omega(2k^2-\ell^2)/8\kappa^4 \quad \text{and} \quad \gamma=-3\omega\ell k/4\kappa^4.$$

Also, the nonlinearity coefficients (as in [9]) are given by $\xi=\omega\kappa^2/2$ and

$$\zeta=\omega(k^5-k^3\ell^2-3k\ell^4-2k^4\kappa+2k^2\ell^2\kappa+2\ell^4\kappa)/2\kappa^2(k-2\kappa).$$

Here k and ℓ are wavenumbers and ω is the wave frequency. They are related by $\omega=\sqrt{g\kappa}$ (the dispersion relation for deep water waves [5]) with g the acceleration due to gravity and κ the wavenumber norm given by $\kappa\equiv\sqrt{k^2+\ell^2}$. For detail description of the formulation of the problem, we refer to the original works [9, 11].

The nonlinear strongly coupled system of equations (1) and (2) will be computed using a fast numerical algorithm based on the spectral method [2, 13] which is explained below.

A. Fourier Spectral Method

We first noticed that by letting $S=A+B$ and $D=A-B$, the system (1) and (2) becomes symmetric, obtained from (1) and (2)

$$i\left(\frac{\partial S}{\partial t}+C_x\frac{\partial S}{\partial x}+C_y\frac{\partial D}{\partial y}\right)+\alpha\frac{\partial^2 S}{\partial x^2}+\beta\frac{\partial^2 S}{\partial y^2}+\gamma\frac{\partial^2 D}{\partial x\partial y}=g(S,D) \tag{3}$$

$$i\left(\frac{\partial D}{\partial t}+C_x\frac{\partial D}{\partial x}+C_y\frac{\partial S}{\partial y}\right)+\alpha\frac{\partial^2 D}{\partial x^2}+\beta\frac{\partial^2 D}{\partial y^2}+\gamma\frac{\partial^2 S}{\partial x\partial y}=G(D,S) \tag{4}$$

where

$$G(u,v)=\frac{1}{8}\left[(\xi+2\eta)\left(|u+v|^2+|u-v|^2\right)u+(\xi-2\eta)\left(|u+v|^2-|u-v|^2\right)v\right] \tag{5}$$

Then, we reduce the above system of PDEs (3)–(4) into a system of ODEs using the Fourier transform of $u(x,y)$ which is defined by

$$\mathcal{F}(u)(k_x,k_y)=\widehat{u}(k_x,k_y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i(k_xx+k_yy)}u(x,y)\,dx\,dy, \tag{6}$$

with the corresponding inverse

$$\mathcal{F}^{-1}(\widehat{u})(x,y) = u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k_x x + k_y y)}\widehat{u}(k_x,k_y)\,dk_x\,dk_y. \tag{7}$$

The function $\widehat{u}(k_x,k_y)$ can be interpreted as the amplitude density of u for wavenumbers k_x, k_y. Now, we take the Fourier transform of both (3) and (4) as

$$i\frac{d\widehat{S}_t}{dt} - \left(k_x C_x + \alpha k_x^2 + \beta k_y^2\right)\widehat{S} - k_y\left(C_y + \gamma k_x\right)\widehat{D} = \widehat{G(S,D)}, \tag{8}$$

$$i\frac{d\widehat{D}_t}{dt} - \left(k_x C_x + \alpha k_x^2 + \beta k_y^2\right)\widehat{D} - k_y\left(C_y + \gamma k_x\right)\widehat{S} = \widehat{G(D,S)}, \tag{9}$$

Letting $k_x C_x + \alpha k_x^2 + \beta k_y^2 = p$ and $k_y\left(C_y + \gamma k_x\right) = r$ (8) and (9) can be written in the matrix form as

$$i\frac{d}{dt}\begin{pmatrix}\widehat{S}\\ \widehat{D}\end{pmatrix} = \begin{pmatrix} p & r\\ r & p\end{pmatrix}\begin{pmatrix}\widehat{S}\\ \widehat{D}\end{pmatrix} + \begin{pmatrix}\widehat{G(S,D)}\\ \widehat{G(D,S}\end{pmatrix} \tag{10}$$

Computing the eigenvalues and eigenvectors the solution to (10) can be written as

$$\begin{aligned}\begin{pmatrix}\widehat{S}\\ \widehat{D}\end{pmatrix} &= \frac{1}{2}\begin{pmatrix} e^{-i\lambda_1 t} + e^{-i\lambda_2 t} & -e^{-i\lambda_1 t} + e^{-i\lambda_2 t}\\ -e^{-i\lambda_1 t} + e^{-i\lambda_2 t} & e^{-i\lambda_1 t} + e^{-i\lambda_2 t}\end{pmatrix}\begin{pmatrix}\widehat{S}(0)\\ \widehat{D}(0)\end{pmatrix}\\ &+ \frac{1}{2}\int_0^t \begin{pmatrix} e^{i\lambda_1\tau}\left(\widehat{G(S,D)} - \widehat{G(D,S)}\right)\\ e^{i\lambda_2\tau}\left(\widehat{G(S,D)} + \widehat{G(D,S)}\right)\end{pmatrix} d\tau\end{aligned} \tag{11}$$

with $\lambda_1 = k_x C_x + \alpha k_x^2 + \beta k_y^2 - k_y\left(C_y + \gamma k_x\right)$ and $\lambda_2 = k_x C_x + \alpha k_x^2 + \beta k_y^2 + k_y\left(C_y + \gamma k_x\right)$.

We exploit the symmetry of the nonlinear function G from (7) in developing a numerical procedure to solve the system of ODEs (10).

B. Spatial discretization (Discrete Fourier Transform)

We discretize the spatial domain $\Omega = [-L/2, L/2]\times[-L/2, L/2]$ into $n\times n$ uniformly spaced grid points $X_{ij} = (x_i, y_j)$ with $\Delta x = \Delta y = L/n$, n even, and L the length of the rectangular mesh Ω. Given $u(X_{ij}) = U_{ij}$, $i,j = 1,2,\cdots,n$, we define the 2D Discrete Fourier transform (2DFT) of u as

$$\widehat{u}_{k_x k_y} = \Delta x\Delta y\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-i(k_x x_i + k_y y_j)}U_{ij}, \quad k_x, k_y = -\frac{n}{2}+1,\cdots,\frac{n}{2} \tag{12}$$

and its inverse 2DFT as

$$U_{ij} = \frac{1}{(2\pi)^2}\sum_{k_x=-n/2+1}^{n/2}\;\sum_{k_y=-n/2+1}^{n/2} e^{i(k_x x_i + k_y y_j)}\widehat{u}_{k_x k_y}, \quad i,j = 1,2,\cdots,n. \tag{13}$$

In equation (12) and (13) the wavenumbers k_x and k_y, and the spatial indexes i and j, take only integer values.

C. Temporal discretization

We solve the initial value problem of the ODE system (10) using the classical fourth order Runge-Kutta (RK4) method and exact treatment for the linear part [2].

Given $t_{\max}$, we discretize the time domain $[0, t_{\max}]$ with equal time steps of size Δt with $t_n = n\Delta t$, $n = 0,1,2,\cdots$, and define $S^n = S(x,y;t_n)$ and $D^n = D(x,y;t_n)$. Initializing $\widehat{S}^n = \widehat{S}(t_n)$ and $\widehat{D}^n = \widehat{D}(t_n)$, we compute the Fourier transforms of the nonlinear terms $\mathcal{F}\left(G\left(\mathcal{F}^{-1}(\widehat{S}^n), \mathcal{F}^{-1}(\widehat{D}^n)\right)\right)$ and $\mathcal{F}\left(G\left(\mathcal{F}^{-1}(\widehat{D}^n), \mathcal{F}^{-1}(\widehat{S}^n)\right)\right)$, and advanced the ODE (10) in time with time step Δt using the explicit RK4 for the nonlinear part, together with an exact solution for the linear part as shown in (11).

D. Simulation setup

The numerical code for the above procedure is implemented in FORTRAN 90 and executed on a Linux cluster. The initial profiles for A and B were taken as Gaussians,

$$A(x, y; 0) = (A_0 + random(O(10^{-3}/\kappa))e^{-\sigma(x^2+y^2)} \tag{14}$$

$$B(x, y; 0) = (B_0 + random(O(10^{-3}/\kappa))e^{-\sigma(x^2+y^2)} \tag{15}$$

In the simulations reported here, we used the parameter values $\theta_0 = \pi/6$, $g = 9.81$, $w = 0.56$, $k = 0.33$, $A_0 = 0.1/\kappa$, $B_0 = A_0, 0$, $\sigma = 1, 0$, $L = 2$ and a grid of 256×256 nodes in the computational domain $[-1, 1] \times [-1, 1]$ with the time step size $\Delta t = 0.01$.

For each simulation we monitor the energies $Q_A(t)$ and $Q_B(t)$, calculated as

$$Q_A(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |A(x, y; t)|^2 \, dx \, dy = \sum_{i=1}^{n} \sum_{j=1}^{n} |A_{ij}|^2 \Delta x \Delta y \tag{16}$$

$$Q_B(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |B(x, y; t)|^2 \, dx \, dy = \sum_{i=1}^{n} \sum_{j=1}^{n} |B_{ij}|^2 \Delta x \Delta y \tag{17}$$

Observing a finite energy will reveal stability of a solution. As soon as the solution becomes unstable, the energy diverges. When the solution dissipates the energy approaches to zero.

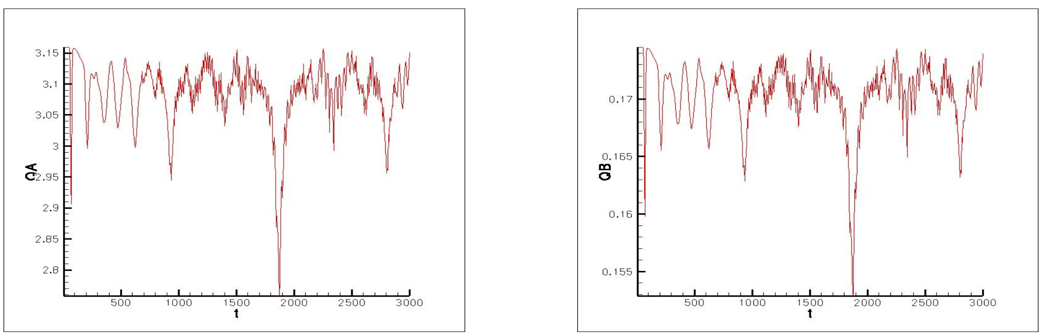

FIG. 1: Energy evolution $Q_A(t)$ (left) and $Q_B(t)$ (right) in a typical 3000 min. simulation.

III. RESULTS

The results presented in this paper represent a preliminary study on dynamics of interacting nonlinear water waves. The problem considered here comprises the dynamics of nonlinear interacting water wave packets through solving the coupled system of equations (1) and (2).

The results that are shown in Fig. 2 are all in dimensional units, where the two interacting waves initially have the amplitude $A = B = 0.1/\kappa + \text{ran}$, with ran representing a random low-amplitude noise, equal to $10^{-3}/\kappa$, in order to enhance instability. The results shown represent different time steps (starting on the left-hand panel and going downwards) for $t = 300/\omega$, $t = 600/\omega$, $t = 900/\omega$ then (right-hand panel) $t = 1200/\omega$, $t = 1500/\omega$; the last figure on the right-hand panel is at the same time as that above it but plotted from a different prospective reflecting the maximum growth rate in the y direction. For our simulations we have taken typical data from ocean waves [4]. The waves A and B in Fig. 2, then have the initial amplitudes $|A| = |B| = 0.1/\kappa \approx 3$ meters. From these figures, we see at $t = 1500/\omega$ (≈ 2680 seconds) that wave A focuses as a localized wave packets with a maximum amplitude of $\approx 0.35/\kappa \approx 10$ meters. We remark for considerable period after the initial step, waves A and B are qualitatively the same (with $|A| > |B|$) before the nonlinear wave-wave interactions set in which results to wave break-up.

In summary, we presented a numerical procedure to solve CNLS equations describing modulational instabilities of a pair of nonlinearly interacting two-dimensional waves in deep water. The simulation results of the full dynamical

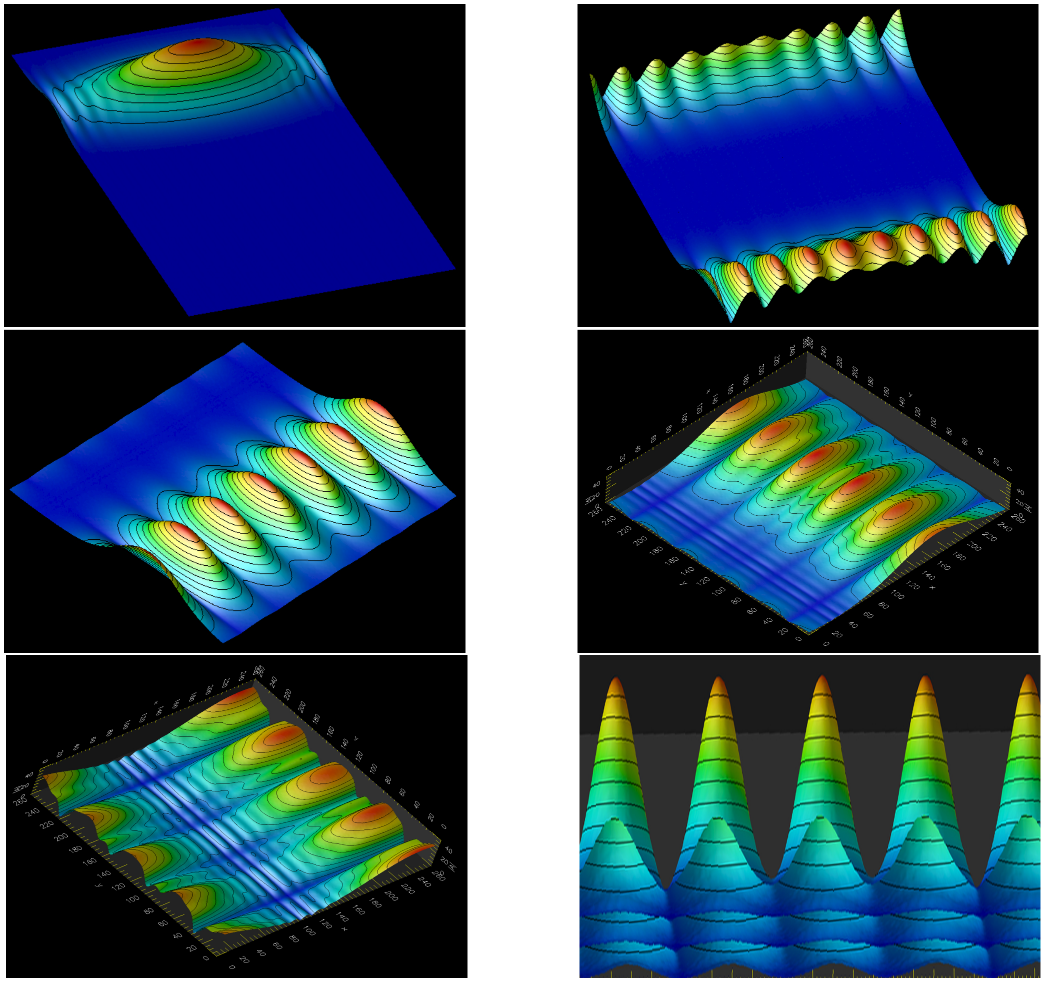

FIG. 2: The interaction between two waves, with equal initial amplitudes $|A| = |B| = 0.1\,\kappa^{-1}$ which are propagating at an angle of $\theta = \pi/6$. A low-amplitude noise equal to $10^{-3}/\kappa$ is added to the initial amplitude in order to enhance the modulation instability.

system reveals that even waves that are separately modulationally stable can, when nonlinear interactions are taken into account, give rise to novel behavior such as the formation of large-amplitude coherent wave packets with amplitudes several times the initial waves. This behavior is quite different from that of a single wave (the case for the original Benjamin-Feir instability) which disintegrates into a wide spectrum of waves. These results are relevant to the nonlinear instability arising from colliding water waves thereby producing large-amplitude oceanic freak waves.

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